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Two-Dimensional Viscous Flow Between Circular Cylinders: Application to Damping Devices for Seismic Isolation

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Abstract—A detailed analysis of the fluid dynamics of the two-dimensional viscous flow between circular cylinders is dealt with in this paper. Analytic solutions are found on the basis of asymptotic expansions with respect to a small parameter defined by the ratio between the difference of the radii and the radius of the internal cylinder. The analysis is related to the study of recently developed devices for seismic isolation of buildings based on modified pile foundation, separated from the soil, in which a viscous fluid is inserted in the void space between the pile and the lining of the surrounding soil. The availability of this analytical solution contributes to obtaining accurate predictions of the force on the pile.

Keywords—Viscous flow, Asymptotic expansion, Seismic isolation.

1. INTRODUCTION

Consider the flow of an incompressible fluid contained between two infinitely long circular cylinders when the inner cylinder is allowed to translate. The flow pattern can be described by the two-dimensional Navier-Stokes equations, which are characterized by three dimensionless parameters: the ratio ε between the difference of cylinders radii and the internal cylinder radius, the ratio α between the maximum displacement of the pile and the radii difference, an apparent Reynolds number R_0.

An approximate solution based on an asymptotic expansion with respect to ε in the limit process for ε tending to zero is proposed in this paper. The analytical solution of the regular perturbation problem we obtained for the streamfunction is computed up to the second order. The corresponding flow is viscosity dominated. Details of the procedure are given in Sections 2 and 3.

Our work is complementary to the numerical study of Duck and Smith [1], who were interested in the influence of a far boundary to the oscillations of a cylinder to justify the difference

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between theoretical results for an oscillating cylinder in an infinite fluid ambient and laboratory measurements. They consequently considered the problem for $\varepsilon \gg 1$.

The motivation of this study lies in a recently proposed system of seismic isolation of buildings [2]. It consists of a foundation characterized by a rigid base standing on foundation piles only partially embedded within the ground and then with the upper side free to move, to be used when the insufficient mechanical properties of soil need the adoption of foundation on piles [3]. However, the increased flexibility of the whole structure leads to larger displacements which are to be limited. In this end, a damping device is needed and the idea of adopting a viscous fluid surrounding the free upper side of the pile and confined by a coaxial cylindrical container was first proposed in [2]. The analysis of this device requires the study of the motion of the fluid filling the zone between the two cylinders. Preliminary studies [2] were primarily concerned with the global response of the structure to seismicly relevant excitation, and so heretofore, flow simplified models have been used. As suggested in these studies, a narrow gap between the cylinders is needed in order to have an effective damping. Our analysis can fulfill the need of a satisfactory prediction of the flow in order to have reliable information for a proper planning of these seismic isolation systems. Moreover, it may have many other applications, such as in the field of lubrification of cylindrical trees. Referring to possible applications, the only limitations of our work concern the nature of fluid and the two-dimensional system geometry.

The analytical solution enables us to consistently determine the most relevant physical quantities, like radial and azimuthal velocities, pressure, and total resistance to the inner cylinder.

A major result is that the resistance to the motion of the internal cylinder is greater than found in previous estimates, in particular, for small values of ε, when the contribution of normal stresses is dominant with respect to skin friction, thus, increasing the effectiveness of the damping system. A comprehensive discussion is given in Section 4.

2. MATHEMATICAL STATEMENT OF THE PROBLEM

Consider the two-dimensional flow of a viscous liquid in the zone between two circular cylinders. The external cylinder, of radius b, is fixed, while the internal one, of radius a, is allowed to move along the x_1-axis. A sketch of the geometry is shown in Figure 1. The dynamics of a viscous fluid is described by the Navier-Stokes equations, that, for an incompressible flow, can be written [4,5] as follows:

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\rho^{-1}\nabla p + \nu \nabla^2 \mathbf{u}, \tag{2}$$

where $\mathbf{u}$ is the velocity, p the pressure, ρ the density, and $\nu = \mu/\rho$ the kinematic viscosity. The flow domain in plane polar coordinates is identified by the variables (r, θ), with $r_0(\theta, t) < r < b$, $0 < \theta < 2\pi$, where

$$r_0(\theta, t) = q(t)\cos\theta + \left(a^2 - q^2(t)\sin^2\theta\right)^{1/2}$$

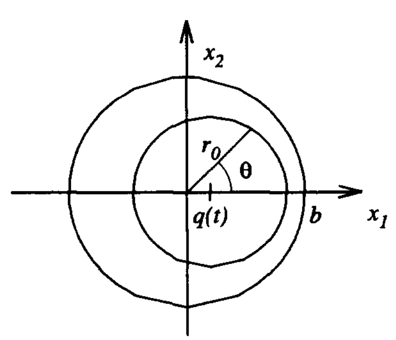

Figure 1. Geometry of the system.

is the polar representation of the inner cylinder and $q(t)$ is its displacement along the x_1 direction. In a two-dimensional incompressible flow, the existence of a potential ψ, such that the velocity is given [4–6] by

$$\mathbf{u} = \nabla^{\perp}\psi = \partial_{x_2}\psi\,\mathbf{e}_1 - \partial_{x_1}\psi\,\mathbf{e}_2 = r^{-1}\partial_\theta\psi\,\mathbf{e}_r - \partial_r\psi\,\mathbf{e}_\theta,$$

may be inferred from (1). As known, the function ψ is constant along each streamline and is called streamfunction.

The evolution equation for the streamfunction is obtained taking the curl of (2) and observing that the vorticity is equal to $-\nabla^2\psi$. In polar coordinates,

$$\partial_t\nabla^2\psi + r^{-1}\partial_\theta\psi\partial_r\nabla^2\psi - r^{-1}\partial_r\psi\partial_\theta\nabla^2\psi = \nu\nabla^2\nabla^2\psi. \tag{3}$$

The mathematical problem is stated linking the above equation to the nonslip boundary conditions

$$\partial_\theta\psi(b,\theta,t) = 0, \tag{4}$$
$$\partial_r\psi(b,\theta,t) = 0, \tag{5}$$
$$\partial_\theta\psi(r_0(\theta,t),\theta,t) = vq'(t)r_0(\theta,t)\cos\theta, \tag{6}$$
$$\partial_r\psi(r_0(\theta,t),\theta,t) = q'(t)\sin\theta. \tag{7}$$

Integrating equations (4) and (6) and using equation (7) yields

$$\psi(b,\theta,t) = 0, \tag{8}$$
$$\psi(r_0(\theta,t),\theta,t) = r_0(\theta,t)q'(t)\sin\theta. \tag{9}$$

Moreover, the problem needs suitable initial conditions.

According to the above statement, the problem is split into two separate parts: the velocity field may be computed from the streamfunction solving equation (3) with boundary conditions (4)–(9) and then the flow may be completely determined computing the pressure field integrating the momentum balance (2).

This procedure will be followed in the next section. In carrying through the analysis, it is useful to operate with dimensionless variables. In the following, the inner cylinder displacement will be represented by

$$q(t) = \Delta f\left(\frac{t}{\tau}\right),$$

where $\Delta < (b-a)$ is the maximum displacement, so that $|f| \le 1$, and τ is a temporal scale of the cylinder motion (e.g., the period for a harmonic oscillation). Physically meaningful dimensionless variables, denoted with an overbar, may be defined as

$$\bar r = \frac{r-a}{b-a}, \qquad \bar t = \frac{t}{\tau}, \qquad \bar\psi = \frac{\psi\tau}{a\Delta}, \qquad \bar p = \frac{p\tau a}{\mu\Delta}, \qquad \bar{\mathbf{u}} = \frac{\tau}{\Delta}\mathbf{u},$$

so that equation (3) is rewritten as follows:

$$\varepsilon^2 R_0\partial_{\bar t}\bar\nabla^2\bar\psi + \varepsilon^2\alpha R_0\left(1+\varepsilon\bar r\right)^{-1}\left(\partial_\theta\bar\psi\partial_{\bar r}\bar\nabla^2\bar\psi - \partial_{\bar r}\bar\psi\partial_\theta\bar\nabla^2\bar\psi\right) = \bar\nabla^2\bar\nabla^2\bar\psi, \tag{10}$$

where

$$\bar\nabla^2 = \partial_{\bar r}^2 + \varepsilon\left(1+\varepsilon\bar r\right)^{-1}\partial_{\bar r} + \varepsilon^2\left(1+\varepsilon\bar r\right)^{-2}\partial_\theta^2,$$

and the boundary conditions become

$$\bar\psi\left(1,\theta,\bar t\right) = 0, \tag{11}$$
$$\partial_{\bar r}\bar\psi\left(1,\theta,\bar t\right) = 0, \tag{12}$$
$$\bar\psi\left(\bar r_0\left(\theta,\bar t\right),\theta,\bar t\right) = f'(t)\left(1+\varepsilon\bar r_0\left(\theta,\bar t\right)\right)\sin\theta, \tag{13}$$
$$\partial_{\bar r}\bar\psi\left(\bar r_0\left(\theta,\bar t\right),\theta,\bar t\right) = \varepsilon f'\left(\bar t\right)\sin\theta. \tag{14}$$

Three parameters arise from the adimensionalization: the ratio $\varepsilon = (b-a)/a$ between the radii difference and the radius of the inner cylinder, the ratio $\alpha = \Delta/(b-a)$ between the maximum displacement of the internal cylinder and the maximum allowable displacement $b-a$, and finally, an apparent Reynolds number $R_0 = a^2/(\tau\nu)$. Comparing (10) with the standard dimensionless form of the incompressible Navier-Stokes equations (see [4]), this flow is characterized by a real Reynolds number $\varepsilon^2\alpha R_0$, as it is natural observing that this is the ratio between the reference length $(b-a)$ of this flow times the reference velocity Δ/τ and the kinematic viscosity ν, and by a Strouhal number α^{-1}.

From $\bar{u}_r = (1+\varepsilon\bar{r})^{-1}\partial_\theta\bar{\psi}$, $\bar{u}_\theta = -\varepsilon^{-1}\partial_{\bar{r}}\bar{\psi}$, the dimensionless version of (2) to be used for the computation of the pressure is then, in plane polar coordinate,

$$\begin{aligned}\partial_{\bar{r}}\bar{p} = {} & \varepsilon^{-1}\partial^2_{\bar{r}\bar{r}}\left((1+\varepsilon\bar{r})^{-1}\partial_\theta\bar{\psi}\right) + (1+\varepsilon\bar{r})^{-1}\partial_{\bar{r}}\left((1+\varepsilon\bar{r})^{-1}\partial_\theta\bar{\psi}\right) + 2(1+\varepsilon\bar{r})^{-1}\partial^2_{\bar{r}\theta}\bar{\psi} \\ & + \varepsilon(1+\varepsilon\bar{r})^{-3}\left(\partial^3_{\theta\theta\theta}\bar{\psi} - \partial_\theta\bar{\psi}\right)\varepsilon R_0(1+\varepsilon\bar{r})^{-1}\partial^2_{\bar{t}\theta}\bar{\psi} \\ & - R_0(1+\varepsilon\bar{r})^{-1}\left[\varepsilon\partial_\theta\bar{\psi}\partial_{\bar{r}}\left((1+\varepsilon\bar{r})^{-1}\partial_\theta\bar{\psi}\right) - \varepsilon(1+\varepsilon\bar{r})^{-1}\partial_{rs}\bar{\psi}\partial^2_{\theta\theta}\bar{\psi} - \left(\partial_{\bar{r}}\bar{\psi}\right)^2\right],\end{aligned} \tag{15}$$

$$\begin{aligned}\partial_\theta\bar{p} = {} & -\varepsilon^{-3}(1+\varepsilon\bar{r})\partial^3_{\bar{r}\bar{r}\bar{r}}\bar{\psi} - \varepsilon^{-2}\partial^2_{\bar{r}\bar{r}}\bar{\psi} + \varepsilon^{-1}(1+\varepsilon\bar{r})^{-1}\left(\partial_{\bar{r}}\bar{\psi} - \partial_{\bar{r}\bar{r}\theta}\bar{\psi}\right) \\ & + 2(1+\varepsilon\bar{r})^{-2}\partial^2_{\theta\theta}\bar{\psi} - R_0\varepsilon^{-1}(1+\varepsilon\bar{r})\partial^2_{\bar{t}\theta}\bar{\psi} + \\ & + \alpha R_0\left[\varepsilon^{-1}\partial_\theta\bar{\psi}\partial^2_{\bar{r}\bar{r}}\bar{\psi} - \varepsilon^{-1}\partial_{\bar{r}}\bar{\psi}\partial^2_{\bar{r}\theta}\bar{\psi} + \partial_\theta\bar{\psi}\partial_{\bar{r}}\bar{\psi}\right].\end{aligned} \tag{16}$$

3. ASYMPTOTIC EXPANSION

In order to study the behaviour of the system for small values of ε, let us consider the asymptotic expansion associated with the limit process for $\varepsilon \to 0^+$, with $\bar{r}$, θ, $\bar{t}$ fixed. Let us write the expansion of the streamfunction as

$$\bar{\psi}(\bar{r},\theta,\bar{t};\varepsilon) = \bar{\psi}_0(\bar{r},\theta,\bar{t}) + \mu_1(\varepsilon)\bar{\psi}_1(\bar{r},\theta,\bar{t}) + o(\mu_1(\varepsilon)), \tag{17}$$

where the gauge function $\mu_1(\varepsilon) = o(1)$ has to be computed from the equation and its boundary conditions [7]. Function $\bar{\psi}_0$ is the basis solution of the problem and $\bar{\psi}_1$ is the first-order correction. Higher-order terms will not be considered in this paper. Note that, $\varepsilon^2\alpha R_0$ being the effective Reynolds number, our limit process is not only a process for small values of the gap between the cylinders, but also for small values of the Reynolds number. So, we know in advance that the corresponding flow will be a viscosity dominated one in which inertial terms on the left-hand side of equation (10) are expected not to give any contribution at the lowest order. Once (17) is substituted into equation (10) for the dimensionless streamfunction, we obtain, reordering all terms,

$$\partial^4_{\bar{r}}\bar{\psi}_0 + 2\varepsilon\partial^3_{\bar{r}}\bar{\psi}_0 + \mu_1(\varepsilon)\partial^4_{\bar{r}}\bar{\psi}_1 + o(\varepsilon,\mu_1(\varepsilon)) = 0. \tag{18}$$

At order $O(1)$, we simply have

$$\partial^4_{\bar{r}}\bar{\psi}_0 = 0. \tag{19}$$

With a look at the boundary conditions (11)–(14), the only way to get a nontrivial first-order correction $\bar{\psi}_1$ and to satisfy equation (18) at $O(\varepsilon)$ is to put $\mu_1(\varepsilon) = \varepsilon$, obtaining

$$\partial^4_{\bar{r}}\bar{\psi}_1 + 2\partial^3_{\bar{r}}\bar{\psi}_0 = 0. \tag{20}$$

Analogously, we expand also the boundary conditions. At the external cylinder, it is straightforward to get from (11),(12),

$$\begin{aligned}\bar{\psi}_0(1,\theta,\bar{t}) &= 0, & \bar{\psi}_1(1,\theta,\bar{t}) &= 0, \\ \partial_{\bar{r}}\bar{\psi}_0(1,\theta,\bar{t}) &= 0, & \partial_{\bar{r}}\bar{\psi}_1(1,\theta,\bar{t}) &= 0.\end{aligned}$$

Conditions (13),(14) have to be applied at the internal border $\bar{r}_0(\theta,\bar{t};\varepsilon)$, that is, on a boundary that depends from ε. In order to have a consistent asymptotic expansion for our solution, with $\bar{\psi}_0$, $\bar{\psi}_1$ independent from ε, we must also expand the boundary [6,7]. Using

$$\bar{r}_0(\theta,\bar{t};\varepsilon) = \alpha f(\bar{t})\cos\theta - \frac{1}{2}\varepsilon\alpha^2 f^2(\bar{t})\sin^2\theta + o(\varepsilon^2),$$

we have

$$\bar{\psi}_0(\alpha f(\bar{t})\cos\theta,\theta,\bar{t}) = f'(\bar{t})\sin\theta, \tag{21}$$
$$\bar{\psi}_1(\alpha f(\bar{t})\cos\theta,\theta,\bar{t}) = \alpha f(\bar{t}) f'(\bar{t})\sin\theta\cos\theta, \tag{22}$$
$$\partial_{\bar{r}}\bar{\psi}_0(\alpha f(\bar{t})\cos\theta,\theta,\bar{t}) = 0, \tag{23}$$
$$\partial_{\bar{r}}\bar{\psi}_1(\alpha f(\bar{t})\cos\theta,\theta,\bar{t}) = f'(\bar{t})\sin\theta + \frac{1}{2}\alpha^2 f^2(\bar{t})\sin^2\theta\,\partial_{\bar{r}}^2\bar{\psi}_0(\alpha f(\bar{t})\cos\theta,\theta,\bar{t}). \tag{24}$$

3.1. Solution of the Perturbative Equations

The solution of (19) for the basis solution $\bar{\psi}_0$ is

$$\bar{\psi}_0(1,\theta,\bar{t}) = \left(A_0(\theta,\bar{t}) + A_1(\theta,\bar{t})\bar{r} + A_2(\theta,\bar{t})\bar{r}^2 + A_3(\theta,\bar{t})\bar{r}^3\right) f'(\bar{t})\sin\theta, \tag{25}$$

where functions $A_i(\theta,\bar{t})$ are determined by the boundary conditions

$$\begin{aligned}
A_0(\theta,\bar{t}) &= (1-3\alpha f(\bar{t})\cos\theta)(1-\alpha f(\bar{t})\cos\theta)^{-3},\\
A_1(\theta,\bar{t}) &= 6\alpha f(\bar{t})\cos\theta(1-\alpha f(\bar{t})\cos\theta)^{-3},\\
A_2(\theta,\bar{t}) &= -3(1+\alpha f(\bar{t})\cos\theta)(1-\alpha f(\bar{t})\cos\theta)^{-3},\\
A_3(\theta,\bar{t}) &= 2(1-\alpha f(\bar{t})\cos\theta)^{-3}.
\end{aligned}$$

The basis flow then resembles a quasi-stationary two-dimensional Poiseuille channel flow, with a modulation in θ and $\bar{t}$ due to the nonuniform boundaries. However, the strong θ-dependence implies a true two-dimensional nonparallel flow with nonzero radial velocity, in particular, for θ close to 0 or π. Inserting this solution into (20), we have for $\bar{\psi}_1$,

$$\bar{\psi}_1(1,\theta,\bar{t}) = \left(B_0(\theta,\bar{t}) + B_1(\theta,\bar{t})\bar{r} + B_2(\theta,\bar{t})\bar{r}^2 + B_3(\theta,\bar{t})\bar{r}^3 - \frac{1}{2}A_3(\theta,\bar{t})\bar{r}^4\right) f'(\bar{t})\sin\theta, \tag{26}$$

where, from the boundary conditions,

$$\begin{aligned}
B_0(\theta,\bar{t}) &= 3\alpha^2 f^2(\bar{t})\cos\theta(\alpha f(\bar{t}) - \cos\theta)(1-\alpha f(\bar{t})\cos\theta)^{-4},\\
B_1(\theta,\bar{t}) &= \left[2\alpha f(\bar{t})\cos\theta\left(1-3\alpha^2 f^2(\bar{t})\right) + 3\alpha^2 f^2(\bar{t})\cos(2\theta) + 1\right](1-\alpha f(\bar{t})\cos\theta)^{-4},\\
B_2(\theta,\bar{t}) &= 3\left[\alpha^2 f^2(\bar{t})\sin^2\theta - \left(1+\alpha f(\bar{t})\cos\theta\right)(1-\alpha^2 f^2(\bar{t}))\right](1-\alpha f(\bar{t})\cos\theta)^{-4},\\
B_3(\theta,\bar{t}) &= 3\left(1-\alpha^2 f^2(\bar{t})\right)(1-\alpha f(\bar{t})\cos\theta)^{-4}.
\end{aligned}$$

The streamlines associated with $\bar{\psi}_0$ and $\bar{\psi}_1$ are shown in Figure 2. For the latter, which takes into account the effects of curvature and shifted boundary, we note two regions of counter rotating flow.

Equations (25) and (26) are a uniformly valid approximation only when very special initial conditions are set, because they require that the initial velocity field itself satisfies these relations. Otherwise, the introduction of a temporal initial boundary layer might be necessary. This is not a real limitation for the applications at the origin of this study, because initial conditions relevant for seismic problems are still fluid with a still cylinder, so we shall not discuss this problem further.

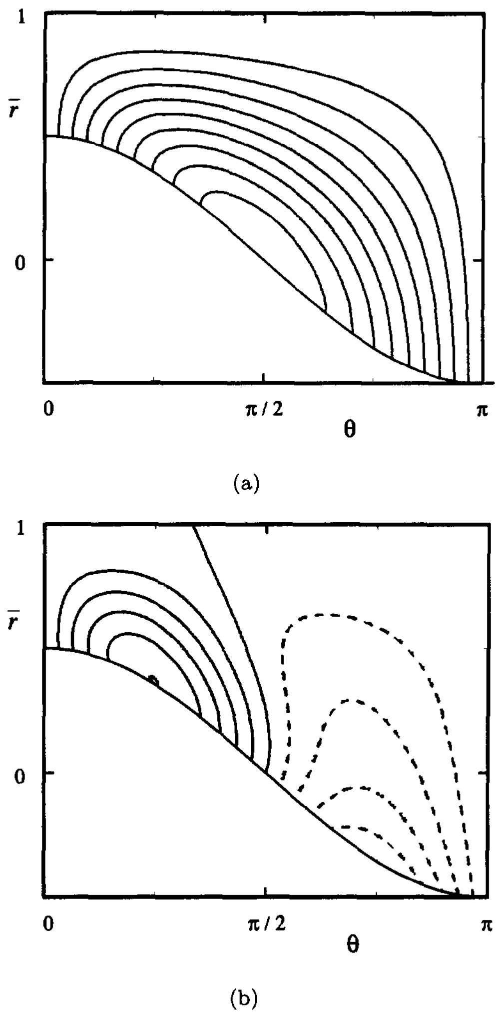

Figure 2. Streamlines of (a) the basis solution $\bar{\psi}_0$ and (b) the first-order correction $\bar{\psi}_1$ for $\alpha f(\bar{t}) = 1/2$. The step between successive curves is $0.1f'(\bar{t})$ for $\bar{\psi}_0$ and $0.05f'(\bar{t})$ for $\bar{\psi}_1$. Negative values are denoted with a dashed line.

3.2. Determination of the Pressure Field

The expansion for $\bar{\psi}$ may be introduced into (15) and (16) in order to compute the pressure field

$$\partial_{\bar{r}}\bar{p} = \varepsilon^{-1}\partial^2_{\bar{r}\theta}\bar{\psi}_0 + O(1), \tag{27}$$

$$\partial_\theta\bar{p} = \varepsilon^{-3}\partial^3_{\bar{r}\bar{r}\bar{r}}\bar{\psi}_0 - \varepsilon^{-2}\left[\partial_{\bar{r}}\left(\bar{r}\partial^2_{\bar{r}\bar{r}}\bar{\psi}_0\right) + \partial^3_{\bar{r}\bar{r}\bar{r}}\bar{\psi}_1\right] + O\left(\varepsilon^{-1}\right). \tag{28}$$

It is evident that, in the limit of $\varepsilon \to 0^+$, no solution for $\bar{p}$ exists. This singular behaviour should be expected in advance, inasmuch as, in this limit, the two bodies come into contact, and an infinite force is needed in order to keep in contact the two bodies embedded in a viscous flow [4,5]. Also, note that the azimuthal component $\bar{u}_\theta$ diverges as ε^{-1}. The same behaviour is also clearly expected for $|\alpha f| \to 1^-$, in which limit, the streamfunction also becomes singular, see (25),(26). Anyway, we may write

$$\bar{p}\left(\bar{r},\theta,\bar{t};\varepsilon\right) = \varepsilon^{-3}\left(\bar{p}_0\left(\bar{r},\theta,\bar{t}\right) + \varepsilon\bar{p}_1\left(\bar{r},\theta,\bar{t}\right) + o\left(\varepsilon\right)\right),$$

where functions $\bar{p}_0$ and $\bar{p}_1$ are given by

$$\begin{aligned} \partial_\theta\bar{p}_0 &= -\partial^3_{\bar{r}\bar{r}\bar{r}}\bar{\psi}_0, & \partial_{\bar{r}}\bar{p}_0 &= 0, \\ \partial_\theta\bar{p}_1 &= -\left[\partial_{\bar{r}}\left(\bar{r}\partial^2_{\bar{r}\bar{r}}\bar{\psi}_0\right) + \partial^3_{\bar{r}\bar{r}\bar{r}}\bar{\psi}_1\right], & \partial_{\bar{r}}\bar{p}_1 &= 0. \end{aligned}$$

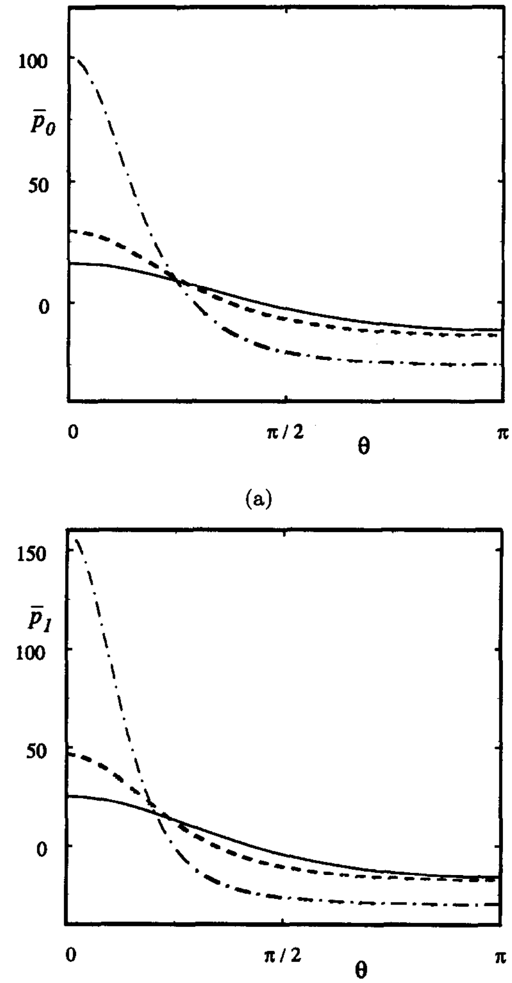

Figure 3. Functions (a) $\bar{p}_0$ and (b) $\bar{p}_1$ for $\alpha f(\bar{t}) = 1/4$ (——), $1/2$ (– – –) and $3/4$ (–·–·–).

Integrating, we have, consequently,

$$\bar{p}_0(\bar{r},\theta,\bar{t}) = 6(\alpha f(\bar{t}))^{-1} f'(\bar{t})(1-\alpha f(\bar{t})\cos\theta)^{-2} + c_0(\bar{t}), \tag{29}$$

$$\begin{aligned}\bar{p}_1(\bar{r},\theta,\bar{t}) &= 3(\alpha f(\bar{t}))^{-1} f'(\bar{t})(1-\alpha f(\bar{t})\cos\theta)^{-2} \\ &\times \left[3 - 2\alpha^2 f^2(\bar{t})\sin^2\theta(1-\alpha f(\bar{t})\cos\theta)^{-1}\right] + c_1(\bar{t}),\end{aligned} \tag{30}$$

where $c_0(\bar{t})$ e $c_1(\bar{t})$ are dummy functions arising from the integration. In Figure 3, they are chosen so that the mean value of the pressure is zero. Note that, for $\alpha f(\bar{t})$ close to unity, that is, when the pile is very close to the external container, pressure variations and, consequently, fluid motion tend to be relevant only in a small region in front of the cylinder, where the gap is far narrower as compared to the other parts, leaving a great region with almost still fluid and constant pressure.

3.3. Force on the Inner Cylinder

The analysis developed above allow us to compute the asymptotic behaviour of the resistance to the cylinder motion, which is one of the most relevant quantities. The force per unit length on the internal cylinder has only a component in the x_1 direction, given by

$$F_1 = \mathbf{e}_1 \cdot \int_0^{2\pi} \mathbf{Tn}\left(r_0^2 + r'^2_0\right)^{1/2} d\theta = \mu\Delta\tau^{-1}\mathbf{e}_1 \cdot \int_0^{2\pi} \bar{\mathbf{T}}\mathbf{n}\left((1+\varepsilon\bar{r}_0)^2 + \varepsilon^2\bar{r}'^2_{02}\right)^{1/2} d\theta,$$

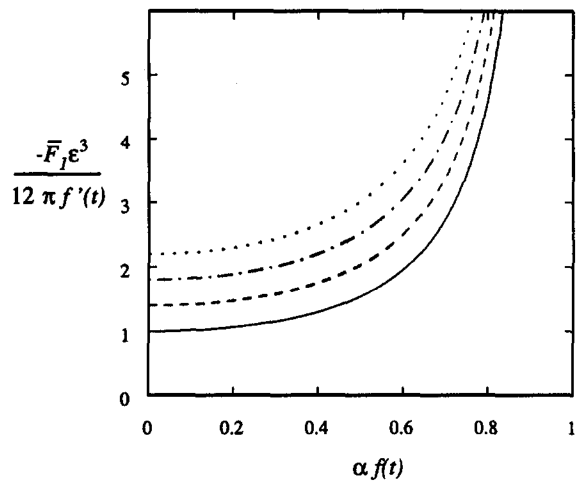

Figure 4. Dimensionless force $\bar{F}_1$ on the inner cylinder following (31). Curves for $\varepsilon = 0$ (——), 0.2 (– – –), 0.4 (–·–·–) and 0.6 (·····) are shown.

where $\mathbf{T}$ is the stress tensor and $\bar{\mathbf{T}} = a\tau(\mu\Delta)^{-1}\mathbf{T}$ is its dimensionless corresponding form, whose components in polar coordinates are

$$\bar{T}_{rr} = -\bar{p} + 2\varepsilon^{-1}\partial_{\bar{r}}\left((1+\varepsilon\bar{r})^{-1}\,\partial_{\bar{r}}\bar{\psi}\right),$$
$$\bar{T}_{r\theta} = -\varepsilon^{-2}\partial^2_{\bar{r}\bar{r}}\bar{\psi} + (1+\varepsilon\bar{r})^{-1}\,\varepsilon^{-1}\partial_{\bar{r}}\bar{\psi} + (1+\varepsilon\bar{r})^{-2}\,\partial_\theta\bar{\psi},$$
$$\bar{T}_{\theta\theta} = -\bar{p} + 2\left[-(1+\varepsilon\bar{r})^{-1}\,\varepsilon^{-1}\partial^2_{\bar{r}\theta}\bar{\psi} + (1+\varepsilon\bar{r})^{-2}\,\partial_\theta\bar{\psi}\right],$$

that admit the following expansion:

$$T_{rr} = -\varepsilon^{-3}\bar{p}_0 - \varepsilon^{-2}\bar{p}_1 + O\left(\varepsilon^{-1}\right),$$
$$T_{r\theta} = -\varepsilon^{-2}\partial^2_{rr}\bar{\psi}_0 + O\left(\varepsilon^{-1}\right),$$
$$T_{\theta\theta} = -\varepsilon^{-3}\bar{p}_0 - \varepsilon^{-2}\bar{p}_1 + O\left(\varepsilon^{-1}\right).$$

Observing that $\mathbf{n} = \mathbf{e}_r + \alpha f(\bar{t})\sin\theta\,\varepsilon\,\mathbf{e}_\theta + O(\varepsilon^2)$, and that pressure is an even function of θ, the dimensionless force per unit length $\bar{F}_1 = F_1\tau/(\mu\Delta)$ is given by

$$\bar{F}_1 = -\varepsilon^{-3}\int_0^{2\pi}\bar{p}_0\cos\theta\,\mathrm{d}\theta - \varepsilon^{-2}\int_0^{2\pi}\left(\bar{p}_1\cos\theta + \alpha f(\bar{t})\,\bar{p}_0\cos(2\theta) - \partial^2_{\bar{r}\bar{r}}\bar{\psi}_0\sin\theta\right)\,\mathrm{d}\theta + O\left(\varepsilon^{-1}\right).$$

Performing all integrations, we finally have

$$\begin{gathered}\bar{F}_1 = -12\pi f'(\bar{t})\left(1-\alpha^2 f^2(\bar{t})\right)^{-3/2}\varepsilon^{-3} \\ \left[1 + 4\varepsilon\left(\alpha f(\bar{t})\right)^{-2}\left(1-\alpha^2 f^2(\bar{t})\right)\left(1-\left(1-\alpha^2 f^2(\bar{t})\right)^{1/2}\right) + O\left(\varepsilon^2\right)\right].\end{gathered} \tag{31}$$

Result (31) shows that, up to the order considered, the resistance is proportional to the instantaneous velocity of the cylinder, given in dimensionless form by $f'(\bar{t})$. Thus, the whole system acts for the pile as a nonlinear "viscous damper", where nonlinearity is referred to the position, and then fit in with the standard introduction of a friction coefficient $C_F = F_1/(\mu q'(t)) = \bar{F}_1/f'(\bar{t})$. It should be noted that this result remains true as long as the nonstationary term $\varepsilon^2\alpha^{-1}\partial_{\bar{t}}\bar{\psi}$ does not enter into the perturbative equations, that is, up to order ε^2.

4. CONCLUDING REMARKS

A perturbative approach is used to study the two-dimensional flow between two cylinders in relative motion, and two terms of an asymptotic expansion for the dimensionless problem in terms of $\varepsilon = (b - a)/a$ are computed for small values of ε. We found that the problem is a regular perturbation one for the dimensionless streamfunction that, however, does not prevent the azimuthal component of velocity and the pressure from having a singular behaviour in the limit for ε going to zero, in a somewhat similar way to the thin airfoil inviscid problem or to the thin shell problem [6,7], as well as, to the analysis of the influence of the Knudsen number in the study of the flow between two rotating cylinders [8].

Up to the order ε, at which we limited our computations, only viscous terms enter in the perturbative equations. On principle, our scheme can be generalized and the solution can be expressed in a formal series of ε. Schemes to be followed are classical in the literature, e.g., [9]. As may be inferred from (10) and (18), all coefficients are polynomials of $\bar{r}$ of increasing order with coefficients function of θ and $\bar{t}$. If convergence were proved, it would provide an analytical solution for this problem. However, a temporal boundary layer is to be expected in general. Anyway, even if the series were not convergent, the first few terms are usually found to be a sufficiently good approximation of the solution for applications. We think it is the case.

Referring to previous studies on the same subject [2], they were based on the assumption of a predominant azimuthal motion (almost correct for small ε), however, working with dimensional variables, they were not able to realize the different order of contribution of the boundary conditions. Consequently, their results were not only limited to the first-order term of our expansions, but also had unnecessary complications that resulted in implicit expressions for the resistance to the cylinder motion. Moreover, they neglected the normal stresses compared to tangential stresses in the computation of the force on the cylinder, which ansatz leads to a resistance of order ε^{-2} instead of ε^{-3} when rewritten in our variables. This brings in a nonnegligible error in applications, where result (31) suggests small values of ε for the use of this system as a damping device.

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